Regime-Switching Jump-Diffusion Processes with Countable Regimes

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Outline

Black-Scholes Model and Extensions

- Suppose the stock price $S(t)$ satisfies the SDE

$$dS(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t)$$

where μ is the expected rate of return and σ is the volatility.

- **Limitations**: μ and σ are assumed to be constant.

Extensions and Modifications:

- $+$ jumps: Kou (2002), Tankov (2011)
- Stochastic Volatility Models: Cui et al. (2018), Heston (1993)
- Black-Scholes Model with Regime-Switching: Barone-Adesi and Whaley (1987), Zhang (2001)

$$dS(t) = \mu(\Lambda(t))S(t)\mathrm{d}t + \sigma(\Lambda(t))S(t)\mathrm{d}W(t)$$

where $\Lambda(\cdot) \in \{1, 2\}$ represents the general market trend: bull or bear.

SIR Model

Kermack and McKendrick (1991a,b)

In the SIR model, a homogeneous host population is subdivided into the following three epidemiologically distinct types of individuals:

(S) The susceptible class: those individuals who are capable of contracting the disease and becoming infected.

(I) The infected class: those individuals who are capable of transmitting the disease to others.

(R) The removed class: infected individuals who are deceased, or have recovered and are either permanently immune or isolated.

If we denote by $S(t)$, $I(t)$, and $R(t)$ the number of individuals at time t in classes (S), (I), and (R), respectively, the spread of infection can be formulated by the following deterministic system:

$$\begin{cases} \mathrm{d}S(t) = (\alpha - \beta S(t)I(t) - \mu S(t))\mathrm{d}t \\ \mathrm{d}I(t) = (\beta S(t)I(t) - (\mu + \rho + \gamma)I(t))\mathrm{d}t \\ \mathrm{d}R(t) = (\gamma I(t) - \mu R(t))\mathrm{d}t \end{cases}$$

where α is the per capita birth rate of the population,
μ is the per capita disease-free death rate,
ρ is the excess per capita death rate of the infective class,
β is the effective per capita contact rate, and
γ is the per capita recovery rate of the infected individuals.

SIR model subject to noises

Tuong et al. (2019)

$$
\begin{cases}
\mathrm{d}S(t) = (-\beta(\Lambda(t))S(t)I(t) + \alpha(\Lambda(t)) - \mu(\Lambda(t))S(t))\mathrm{d}t \\
\qquad\qquad\qquad\qquad\qquad\qquad -\widetilde{\beta}(\Lambda(t))S(t)I(t)\mathrm{d}B(t) \\
\mathrm{d}I(t) = (\beta(\Lambda(t))S(t)I(t) - (\mu(\Lambda(t)) + \rho(\Lambda(t)) + \gamma(\Lambda(t)))I(t))\mathrm{d}t \\
\qquad\qquad\qquad\qquad\qquad\qquad +\widetilde{\beta}(\Lambda(t))S(t)I(t)\mathrm{d}B(t) \\
\mathrm{d}R(t) = (\gamma(\Lambda(t))I(t) - \mu(\Lambda(t))R(t))\mathrm{d}t
\end{cases}
$$

where $\Lambda(t)$ takes value in a discrete set (e.g., a continuous-time Markov chain), modeling the abrupt changes of the parameters.

The rationale: Nonpharmaceutical interventions (such as social distancing, school closures, travel restrictions, etc.) as well the pharmaceutical interventions (such as vaccination), may result in sudden instantaneous transitions between two or more sets of parameter values of the model in different regimes.

Motivations

- More such examples in financial engineering, manufacturing systems, wireless communication networks, social networks, ecological modeling, inventory control, etc.
- Fluctuations, jumps, and structural changes.
- Continuous dynamics and discrete events coexist.
- The traditional differential equation setup is inadequate.
- Regime-switching jump diffusion provides a unified and convenient framework for these complicated applications.
- Mathematically interesting.

Regime-Switching Jump Diffusion

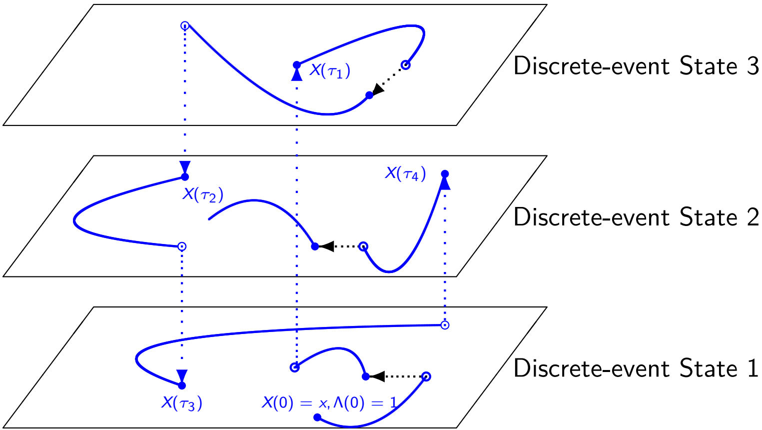

Figure: A "Sample Path" of a Regime-Switching Jump Diffusion.

Regime-Switching Jump Diffusion

Mathematical Description

- Let $(X, \Lambda) \subset \mathbb{R}^d \times \mathbb{S}$ be a strong Markov process with càdlàg paths, where $d \in \mathbb{Z}_+$ and $\mathbb{S} := \{1, 2, \cdots\}$.
- The first component $X \in \mathbb{R}^d$ satisfies the following SDE

$$\begin{aligned} \mathrm{d}X(t) &= \sigma(X(t), \Lambda(t))\mathrm{d}W(t) + b(X(t), \Lambda(t))\mathrm{d}t \\ &\quad + \int_U c(X(t-), \Lambda(t-), u)\widetilde{N}(\mathrm{d}t, \mathrm{d}u) \end{aligned} \tag{1}$$

- The second component $\Lambda \in \mathbb{S} = \{1, 2, \dots\}$ satisfies

$$\Lambda(t) = \Lambda(0) + \int_0^t \int_{\mathbb{R}_+} h(X(s-), \Lambda(s-), r) N_1(\mathrm{d}s, \mathrm{d}r). \tag{2}$$

- $\sigma(x, k) \in \mathbb{R}^{d\times d}$, $b(x, k), c(x, k, u) \in \mathbb{R}^d$ for $x \in \mathbb{R}^d$, $k \in \mathbb{S}$ and $u \in U$.
- $\big(U, \mathcal{B}(U), \nu\big)$ is a measurable space, ν is a σ-finite measure on $(U, \mathcal{B}(U))$.
- W is a d-dimensional Brownian motion.
- $N(\mathrm{d}t, \mathrm{d}u)$ is a Poisson random measure with characteristic measure ν corresponding to a Poisson point process $p(t)$.
- $\widetilde{N}(\mathrm{d}t, \mathrm{d}u) = N(\mathrm{d}t, \mathrm{d}u) - \nu(\mathrm{d}u)\mathrm{d}t$ is the compensated Poisson random measure on $[0, \infty) \times U$.
- N_1 is a Poisson random measure on $[0, \infty) \times \mathbb{R}_+$ with characteristic measure λ (the Lebesgue measure).
- p, W, and N_1 are independent.

- The rate matrix $Q(x) = (q_{kl}(x))$ satisfies

$$\begin{aligned} 0 \le q_{kl}(x) &< +\infty && \text{for} \quad k \neq l \\ q_k(x) := -q_{kk}(x) &< +\infty && \text{(stable)} \\ q_k(x) = \sum_{l \in \mathbb{S} \setminus \{k\}} q_{kl}(x) & && \text{(conservative)} \end{aligned}$$

- For each $x \in \mathbb{R}^n$, let $\{\Delta_{ij}(x) : i, j \in \mathbb{S}\}$ be a family of consecutive (with respect to the lexicographic ordering on $\mathbb{S} \times \mathbb{S}$) and disjoint intervals on $[0, \infty)$, and the length of the interval $\Delta_{ij}(x)$ is equal to $q_{ij}(x)$.
- We then define a function h: $\mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}$ by

$$h(x, k, r) = \sum_{l \in \mathbb{S}} (l - k) \mathbf{1}_{\Delta_{kl}(x)}(r).$$

  Note that for each $x \in \mathbb{R}^d$ and $k \in \mathbb{S}$, $h(x, k, r) = l - k$ if $r \in \Delta_{kl}(x)$ for some $l \neq k$; otherwise $h(x, k, r) = 0$.

Related Work

1. Cloez, B. and Hairer, M. (2015). Exponential ergodicity for Markov processes with random switching. *Bernoulli*, 21(1):505–536.
2. Mao, X. and Yuan, C. (2006). *Stochastic differential equations with Markovian switching*. Imperial College Press, London.
3. Nguyen, D.H. and Yin, G. (2018), Stability of regime-switching diffusion systems with discrete states belonging to a countable set, *SIAM J. Control Optim.*, 56 (5), 3893–3917.
4. Shao, J. (2015). Strong solutions and strong Feller properties for regime-switching diffusion processes in an infinite state space *SIAM J. Control Optim.*, 53(4): 2462–2479.
5. Shao, J. and Yuan, C. (2019). Stability of regime-switching processes under perturbation of transition rate matrices, *Nonlinear Anal. Hybrid Syst.*, 33 , 211–226.
6. Xi, F. (2009). Asymptotic properties of jump-diffusion processes with state-dependent switching. *Stochastic Process. Appl.*, 119(7):2198–2221.
7. Yang, H. and Li, X. (2018). Explicit approximations for nonlinear switching diffusion systems in finite and infinite horizons. *Journal of Differential Equations.* 265(7), 2921-2967.
8. Yin, G. G. and Zhu, C. (2010). *Hybrid Switching Diffusions: Properties and Applications*, Springer, New York.
9. ...

In this work

- Solution: Existence and Uniqueness
  - Coefficients not necessarily Lipschitz
  - Infinite number of switching states
- Asymptotic Property: Ergodicity
  - Invariant measure: existence and uniqueness
  - Convergence rate: exponential ergodicity

Growth Condition I

Suppose there exists a nondecreasing function $\zeta : [0, \infty) \mapsto [1, \infty)$ that is continuously differentiable and satisfies

$$\int_0^\infty \frac{\mathrm{d}r}{r\zeta(r) + 1} = \infty,$$

such that for some $\kappa > 0$ and all $x \in \mathbb{R}^d$,

$$2\langle x, b(x, k)\rangle + |\sigma(x, k)|^2 + \int_U |c(x, k, u)|^2 \nu(\mathrm{d}u) \leq \kappa[|x|^2 \zeta(|x|^2) + 1],$$

Examples: $\zeta(r) = 1$ and $\zeta(r) = \log r$ or $\log r \log(\log r)$ for r large.

Growth Condition II

$Q(x) = (q_{kl}(x))$ satisfies

$$q_k(x) := \sum_{l \in \mathbb{S}} q_{kl}(x) \le Hk,$$

$$\sum_{l \in \mathbb{S}} q_{kl}(x)(f(l) - f(k))^2 \le H(\Phi(x) + f(k) + 1),$$

for all $x, y \in \mathbb{R}^d$ and $k, l \in \mathbb{S}$, where

$$\Phi(x) := \exp\left\{ \int_0^{|x|^2} \frac{\mathrm{d}z}{z\zeta(z) + 1} \right\}, \quad x \in \mathbb{R}^d$$

and $f : \mathbb{S} \mapsto \mathbb{R}_+$ is nondecreasing and satisfies $\lim_{m \to \infty} f(m) = \infty$.

Local Non-Lipschitz Condition I

- For some $H > 0$ and $\delta \in (0, 1]$,

$$\sum_{l \in \mathbb{S} \setminus \{k\}} |q_{kl}(x) - q_{kl}(y)| \leq H|x - y|^{\delta}.$$

- If $d \geq 2$, then there exist a $\delta_0 > 0$ and a nondecreasing and concave function $\rho : [0, \infty) \mapsto [0, \infty)$ satisfying

$$0 < \frac{\rho(r)}{(1+r)^2} \leq \rho(r/(1+r)), \ \forall r > 0, \text{ and } \int_{0+} \frac{\mathrm{d}r}{\rho(r)} = \infty,$$

s.t. for all $R > 0$ and $x, z \in \mathbb{R}^d$ with $|x| \vee |z| \leq R$ and $|x - z| \leq \delta_0$, there exists a $\kappa_R > 0$ s.t.

$$2\langle x - z, \ b(x, k) - b(z, k)\rangle + |\sigma(x, k) - \sigma(z, k)|^2 + \int_U |c(x, k, u) - c(z, k, u)|^2 \nu(\mathrm{d}u) \leq \kappa_R \rho(|x - z|^2).$$

Examples: $\rho(r) = r$, $\rho(r) = r\log(1/r)$, $\rho(r) = r\log(\log(1/r))$ and $\rho(r) = r\log(1/r)\log(\log(1/r))$ for $r \in (0, \delta)$ with $\delta > 0$ small.

Local Non-Lipschitz Condition II

- If $d = 1$, then $\exists \delta_0 > 0$ and a nondecreasing and concave function $\rho : [0, \infty) \mapsto [0, \infty)$ satisfying $\int_{0+} \frac{\mathrm{d}r}{\rho(r)} = \infty$ s.t. for all $R > 0$, $x, z \in \mathbb{R}$ with $|x| \vee |z| \leq R$ and $|x - z| \leq \delta_0$,

  $$\operatorname{sgn}(x - z)(b(x, k) - b(z, k)) \leq \kappa_R \rho(|x - z|),$$
  $$|\sigma(x, k) - \sigma(z, k)|^2 + \int_U |c(x, k, u) - c(z, k, u)|^2 \nu(\mathrm{d}u) \leq \kappa_R |x - z|,$$

  where $\kappa_R > 0$ and $\operatorname{sgn}(a) = I_{\{a>0\}} - I_{\{a\leq 0\}}$. In addition, either

  the function $x \mapsto x + c(x, k, u)$ is nondecreasing for all $u \in U$;

  or, there exists some $\beta > 0$ s.t.

  $$|x - z + \theta(c(x, k, u) - c(z, k, u))| \geq \beta |x - z|$$

  for all $(x, z, u, \theta) \in \mathbb{R} \times \mathbb{R} \times U \times [0, 1]$.

Regime-Switching Jump Diffusion: Existence and Uniqueness

$(X, \Lambda) \in \mathbb{R}^d \times \{1, 2, \dots\}$ satisfies

$$\begin{aligned} \mathrm{d}X(t) &= \sigma(X(t), \Lambda(t))\mathrm{d}W(t) + b(X(t), \Lambda(t))\mathrm{d}t \\ &\quad + \int_U c(X(t-), \Lambda(t-), u)\widetilde{N}(\mathrm{d}t, \mathrm{d}u) \end{aligned} \tag{1}$$

$$\mathrm{d}\Lambda(t) = \int_{\mathbb{R}_+} h(X(t-), \Lambda(t-), r) N_1(\mathrm{d}t, \mathrm{d}r). \tag{2}$$

Theorem 1

Under the above Assumptions, for each $(x, k) \in \mathbb{R}^d \times \mathbb{S}$, the system (1)–(2) has a unique non-explosive strong solution $(X(t), \Lambda(t))$ with initial condition $(X(0), \Lambda(0)) = (x, k)$.

The Infinitesimal Generator

The infinitesimal generator of (X, Λ) is given by

$$\mathcal{A}f(x,k) := \mathcal{L}_k f(x,k) + Q(x)f(x,k), \tag{3}$$

where

$$\begin{aligned}
\mathcal{L}_k f(x,k) &:= \frac{1}{2}\mathrm{tr}\big(a(x,k)\nabla^2 f(x,k)\big) + \langle b(x,k), \nabla f(x,k)\rangle \\
&\quad + \int_U \big(f(x + c(x,k,u),k) - f(x,k) \\
&\qquad - \langle \nabla f(x,k), c(x,k,u)\rangle\big)\nu(\mathrm{d}u), \\
Q(x)f(x,k) &:= \sum_{j\in\mathbb{S}} q_{kj}(x)[f(x,j) - f(x,k)].
\end{aligned}$$

The martingale problem

For any $(x,k) \in \mathbb{R}^d \times \mathbb{S}$, the martingale problem for the operator $\mathcal{A}$ has a unique solution $\mathbb{P}^{(x,k)} \in \mathscr{P}(\Omega)$ starting from (x,k):

- $\mathbb{P}^{(x,k)}\{(X(0),\Lambda(0)) = (x,k)\} = 1$,
- for each $f \in C_c^\infty(\mathbb{R}^d \times \mathbb{S})$,

$$M_t^{(f)} := f(X(t),\Lambda(t)) - f(X(0),\Lambda(0)) - \int_0^t \mathcal{A}f(X(s),\Lambda(s))\mathrm{d}s$$

is an $\{\mathcal{F}_t\}$-martingale with respect to $\mathbb{P}^{(x,k)}$, where (X,Λ) is the coordinate process on $\Omega := D([0,\infty),\mathbb{R}^d \times \mathbb{S})$:

$$(X(t,\omega),\Lambda(t,\omega)) = \omega(t) \in \mathbb{R}^d \times \mathbb{S}, \quad t \geq 0, \omega \in \Omega.$$

From this point on, we assume that the martingale problem for $\mathcal{A}$ is well-posed.

Define

$$P_t f(x,k) := \mathbb{E}[f(X^{(x,k)}(t), \Lambda^{(x,k)}(t))], \quad \forall f \in \mathcal{B}_b(\mathbb{R}^d \times \mathbb{S})$$

and any $\mu \in \mathscr{P}(\mathbb{R}^d \times \mathbb{S})$

$$\mu P_t(A) := \int_{\mathbb{R}^d \times \mathbb{S}} P(t,(x,k),A)\mu(\mathrm{d}x, \mathrm{d}k), \quad \forall A \in \mathcal{B}(\mathbb{R}^d \times \mathbb{S})$$

Overview of Ergodic Theory

A σ-finite measure $\pi(\cdot)$ on $\mathcal{B}(\mathbb{R}^d \times \mathbb{S})$ is called *invariant* for the semigroup P_t if for any $t \geq 0$

$$\pi(A) = \pi P_t(A), \ \forall A \in \mathcal{B}(\mathbb{R}^d \times \mathbb{S}).$$

- Feller property + tightness $\Longrightarrow$ invariant probability measure exists.
- strong Feller + irreducibility $\Longrightarrow$ at most one invariant measure.
- Foster-Lyapunov condition + petiteness $\Longrightarrow$ exponential ergodicity.

- We say that the semigroup P_t or the process (X, Λ) is
  - *Feller continuous* if $P_t f \in C_b(\mathbb{R}^d \times \mathbb{S})$ for all $t \geq 0$ and $\lim_{t \downarrow 0} P_t f(x, k) = f(x, k)$ for all $f \in C_b(\mathbb{R}^d \times \mathbb{S})$ and $(x, k) \in \mathbb{R}^d \times \mathbb{S}$,
  - *strong Feller continuous* if $P_t f \in C_b(\mathbb{R}^d \times \mathbb{S})$ for every $f \in \mathcal{B}_b(\mathbb{R}^d \times \mathbb{S})$ and $t > 0$.
- Goal: To find sufficient conditions for Feller and strong Feller properties.

Feller Property

Assumption 1

- For each $k \in \mathbb{S}$, the coefficients $b(\cdot, k), \sigma(\cdot, k), c(\cdot, k, \cdot)$ satisfy the local non-Lipschitz conditions of Theorem 1.
- For each $k \in \mathbb{S}$, there exists a concave function $\gamma_k : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $\gamma_k(0) = 0$ s.t. for all $x, y \in \mathbb{R}^d$ with $|x| \vee |y| \le R$,

$$\sum_{l \in \mathbb{S} \setminus \{k\}} |q_{kl}(x) - q_{kl}(y)| \le \kappa_R \gamma_k(|x - y|), \quad \kappa_R > 0.$$

- The martingale problem for $\mathcal{A}$ is well-posed.

Theorem 2

Under Assumption 1, the process (X, Λ) has Feller property.

Strong Feller Property: Assumptions

Assumption 2

(i) For each $k \in \mathbb{S}$, there exists a concave function $\gamma_k : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $\gamma_k(0) = 0$ s.t. for all $x, y \in \mathbb{R}^d$ with $|x| \vee |y| \leq R$,

$$\sum_{l \in \mathbb{S} \setminus \{k\}} |q_{kl}(x) - q_{kl}(y)| \leq \kappa_R \gamma_k(|x - y|), \quad \kappa_R > 0.$$

(ii) For every $R > 0$ there exits a constant $\lambda_R > 0$ such that

$$\langle \xi, a(x,k)\xi \rangle \geq \lambda_R |\xi|^2, \qquad \xi \in \mathbb{R}^d,$$

for all $x \in \mathbb{R}^d$ with $|x| \leq R$, where $a(x,k) := \sigma(x,k)\sigma(x,k)^T$.

Strong Feller Property: Assumptions (cont'd)

(iii) $\exists \delta_0 > 0$ and a nonnegative function $g \in C(0, \infty)$ satisfying $\int_0^1 g(r)dr < \infty$, s.t. for each $R > 0$, there exists a constant $\kappa_R > 0$ so that either (a) or (b) below holds:

(a) If $d = 1$, then

$$2(x-z)(b(x,k) - b(z,k)) + \int_U |c(x,k,u) - c(z,k,u)|^2 \nu(\mathrm{d}u) \le 2\kappa_R |x-z| g(|x-z|),$$

for all $x, z \in \mathbb{R}$ with $|x| \vee |z| \le R$ and $|x - z| \le \delta_0$.

(b) If $d \ge 2$, then

$$|\sigma_{\lambda_R}(x,k) - \sigma_{\lambda_R}(z,k)|^2 + 2\langle x-z, b(x,k) - b(z,k)\rangle + \int_U |c(x,k,u) - c(z,k,u)|^2 \nu(\mathrm{d}u) \le 2\kappa_R |x-z| g(|x-z|),$$

for all $x, z \in \mathbb{R}^d$ with $|x| \vee |z| \le R$ and $|x - z| \le \delta_0$, where σ_{λ_R} is the unique symmetric nonnegative definite matrix-valued function such that $\sigma^2_{\lambda_R}(x,k) = a(x,k) - \lambda_R I$.

Strong Feller Property

Theorem 3

Under Assumption 2, the process (X, Λ) *has strong Feller property.*

- Assumption 2 places very mild conditions on the coefficients. It allows us to treat, for example, the case of Hölder continuous coefficients by taking $g(r) = r^{-p}$ for $0 \leq p < 1$.
- In particular, for the case when $d = 1$, only the drift and the jump coefficients are required to satisfy the regularity conditions.
- Such conditions were motivated by Priola and Wang (2006).

Irreducibility

- Define

$$P(t,(x,k),B\times\{l\}) := P_t 1_{B\times\{l\}}(x,k)$$
$$= \mathbb{P}\{(X(t),\Lambda(t))\in B\times\{l\} | (X(0),\Lambda(0))=(x,k)\},$$

  for $B\in\mathcal{B}(\mathbb{R}^d)$ and $l\in\mathbb{S}$.
- The semigroup P_t is said to be *irreducible* if for any $t>0$ and $(x,k)\in\mathbb{R}^d\times\mathbb{S}$, we have

$$P(t,(x,k),B\times\{l\})>0$$

  for all $l\in\mathbb{S}$ and all nonempty open set $B\in\mathcal{B}(\mathbb{R}^d)$.
- Irreducibility is a topological property of the underlying process. Roughly speaking, irreducibility says that every point is reachable from any other point in the state space.

Irreducibility: Assumptions

Assumption 3

(i) Assumption 2 (iii) holds.

(ii) For any $x \in \mathbb{R}^d$ and $k \in \mathbb{S}$, we have

$$2|\langle x, b(x,k)\rangle| + |\sigma(x,k)|^2 + \int_U |c(x,k,u)|^2 \nu(\mathrm{d}u) \le \kappa(|x|^2 + 1),$$

and $\langle \xi, a(x,k)\xi\rangle \ge \lambda|\xi|^2$ for all $\xi \in \mathbb{R}^d$

(iii) There exists a positive constant κ_0 such that $0 \le q_{kl}(x) \le \kappa_0 l 3^{-l}$ for all $x \in \mathbb{R}^d$ and $k \ne l \in \mathbb{S}$.

(iv) For any $k, l \in \mathbb{S}$, there exist $k_0, k_1, ..., k_n \in \mathbb{S}$ with $k_i \ne k_{i+1}, k_0 = k$, and $k_n = l$ such that the set $\{x \in \mathbb{R}^d : q_{k_i k_{i+1}}(x) > 0\}$ has positive Lebesgue measure for all $i = 0, 1, \ldots, n-1$.

Irreducibility

Theorem 4

Suppose that Assumption 3 holds. Then the semigroup P_t is irreducible.

Exponential Ergodicity

For any positive function $f\colon \mathbb{R}^d \times \mathbb{S} \mapsto [1,\infty)$ and any signed measure ν defined on $\mathcal{B}(\mathbb{R}^d \times \mathbb{S})$, we write

$$\|\nu\|_f := \sup\{|\nu(g)| : g \in \mathcal{B}(\mathbb{R}^d \times \mathbb{S}) \text{ satisfying } |g| \leq f\},$$

where $\nu(g) := \sum_{l\in\mathbb{S}} \int_{\mathbb{R}^d} g(x,l)\nu(\mathrm{d}x, l)$.

For a function $\infty > f \geq 1$ on $\mathbb{R}^d \times \mathbb{S}$, the process (X, Λ) is said to be *f-exponentially ergodic* if there exists a probability measure $\pi(\cdot)$, a constant $\theta \in (0,1)$ and a finite-valued function $\Theta(x,k)$ such that

$$\|P(t,(x,k),\cdot) - \pi(\cdot)\|_f \leq \Theta(x,k)\theta^t \tag{4}$$

for all $t \geq 0$ and all $(x,k) \in \mathbb{R}^d \times \mathbb{S}$.

Harris' Theorem

Suppose

- all compact subsets of $\mathbb{R}^d \times \mathbb{S}$ are *petite* for some skeleton chain of $(X(t), \Lambda(t))$, and
- there exists a Foster-Lyapunov function $U : \mathbb{R}^d \times \mathbb{S} \to [0, \infty)$; that is, U satisfies
  (i) $U(x, k) \to \infty$ as $|x| \vee k \to \infty$,
  (ii) $\mathcal{A}U(x, k) \le -\alpha U(x, k) + \beta$ for all $x \in \mathbb{R}^d$, $k \in \mathbb{S}$, where α, β are positive constants.

then the process (X, Λ) is f-exponentially ergodic with $f(x, k) := U(x, k) + 1$ and $\Theta(x, k) = B(U(x, k) + 1)$, where B is a finite constant.

- By Meyn and Tweedie (1992), if P_t is Feller and irreducible, then all compact subsets of $\mathbb{R}^d \times \mathbb{S}$ are petite for the h-skeleton chain.
- Question: How to verify the Foster-Lyapunov drift condition?

[1]

[1] A set $B \in \mathcal{B}(\mathbb{R}^d \times \mathbb{S})$ and a sub-probability measure φ on $\mathcal{B}(\mathbb{R}^d \times \mathbb{S})$ are called *petite* for the h-skeleton chain $\{(X(nh), \Lambda(nh)), n = 0, 1, 2, \dots\}$ $(h > 0)$ if for some $a \in \mathscr{P}(\mathbb{Z}_+)$, we have

$$K_a((x,k),\cdot) := \sum_{i=1}^{\infty} a(i) P(ih, (x,k), \cdot) \geq \varphi(\cdot) \quad \text{for all } (x,k) \in B.$$

Exponential Ergodicity: Assumptions

Assumption 4

(a) There exists an increasing function $\phi : \mathbb{S} \to [0, \infty)$ such that $\lim_{k\to\infty} \phi(k) = \infty$ and

$$\sum_{j\in\mathbb{S}} q_{kj}(x)\,[\phi(j) - \phi(k)] \le C_1 - C_2\phi(k) \quad \text{for all} \;\; k \in \mathbb{S}, x \in \mathbb{R}^d,$$

where $C_1 \ge 0$ and $C_2 > 0$ are constants.

(b) There exists a bounded and x-independent q-matrix $\hat{Q} = (\hat{q}_{ij})$ which is strongly exponentially ergodic with invariant measure $\nu = (\nu_1, \nu_2, ...)$ such that

$$\sup_{k\in\mathbb{S}} \sum_{j\in\mathbb{S}} |q_{kj}(x) - \hat{q}_{kj}| \to 0 \;\; \text{as} \;\; x \to \infty.$$

Exponential Ergodicity

Assumption 4 (cont'd)

(c) There exists a twice continuously differentiable and norm-like function $V : \mathbb{R}^d \to [1, \infty)$ such that for each $k \in \mathbb{S}$

$$\mathcal{L}_k V(x) \leq \alpha_k V(x) + \beta_k \quad \text{for all } x \in \mathbb{R}^d,$$

where $\{\alpha_k\}_{k \in \mathbb{S}}$ and $\{\beta_k\}_{k \in \mathbb{S}}$ are bounded sequences of real numbers such that $\beta_k \geq 0$ and

$$\sum_{k \in \mathbb{S}} \alpha_k \nu_k < 0.$$

Remark: Similar conditions for stability in Nguyen and Yin (2018), Shao and Xi (2014).

Theorem 5

Suppose Assumptions 2, 3, and 4 holds. Then the process (X, Λ) *is exponentially ergodic.*

Example

$$
\begin{aligned}
\mathrm{d}X(t) = {} & b(X(t),\Lambda(t))\mathrm{d}t + \sigma(X(t),\Lambda(t))\mathrm{d}W(t) \\
& + \int_U c(X(t-),\Lambda(t-),u)\widetilde{N}(\mathrm{d}t,\mathrm{d}u),
\end{aligned} \tag{5}
$$

where $W \in \mathbb{R}^2$ is a Brownian motion, $\Lambda \subset \mathbb{S} := \{1, 2, ...\}$, $\widetilde{N}$ is the compensated Poisson random measure on $[0,\infty) \times U$ with intensity $\mathrm{d}t\frac{\mathrm{d}u}{|u|^{2+\delta}}$ for some $\delta \in (0,2)$, where $U = \{u \in \mathbb{R}^2 : 0 < |u| < 1\}$, and

$$
\sigma(x,k) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad b(x,k) = \begin{cases} -x & \text{if} \quad k = 1 \\ \frac{1}{4k}x & \text{if} \quad k \geq 2 \end{cases},
$$

$$
q_{kj}(x) = \begin{cases} \frac{1}{2}\frac{k}{k+e^{-|x|^2}} & \text{if} \quad j = 1, k \neq j, \\ \frac{1}{3^{j-1}}\frac{k}{k+e^{-|x|^2}} & \text{if} \quad j > 1, k \neq j, \end{cases}
$$

$$
c(x,k,u) = \gamma\frac{1}{\sqrt{2k}}|u|x, \text{ where } \gamma > 0 \text{ satisfies } \gamma^2\int_U |u|^2\nu(du) = 1.
$$

We can show that (5) is exponentially ergodic by verifying the assumptions of Theorem 5.

Finally

Thank you!

References

1. Xi, F. & Zhu, C. (2017). On Feller and strong Feller properties and exponential ergodicity of regime-switching jump diffusion processes with countable regimes. *SIAM J. Control Optim.*, *55*(3), 1789–1818.
2. Xi, F., Yin, G., & Zhu, C. (2019). Regime-switching jump diffusions with non-Lipschitz coefficients and countably many switching states: existence and uniqueness, Feller, and strong Feller properties. In *Modeling, stochastic control, optimization, and applications*, volume 164 of *IMA Vol. Math. Appl.* (pp. 571–599). Springer, Cham.
3. Kunwai, K. & Zhu, C. (2020). On Feller and Strong Feller Properties and Irreducibility of Regime-Switching Jump Diffusion Processes with Countable Regimes. *Nonlinear Analysis: Hybrid Systems*, 38, 100946.

Barone-Adesi, G. and Whaley, R. E. (1987). Efficient analytic approximation of American option values. *J. Finance*, 42(2):301–320.

Cui, Z., Kirkby, J. L., and Nguyen, D. (2018). A general valuation framework for sabr and stochastic local volatility models. *SIAM Journal on Financial Mathematics*, 9(2):520–563.

Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Rev. Financ. Stud.*, 6(2):327–343.

Kermack, W. O. and McKendrick, A. G. (1991a). Contributions to the mathematical theory of epidemics–I. *Bulletin of Mathematical Biology*, 53(1-2):33–55.

Kermack, W. O. and McKendrick, A. G. (1991b). Contributions to the mathematical theory of epidemics–II. the problem of endemicity. *Bulletin of Mathematical Biology*, 53(1-2):57–87.

Kou, S. G. (2002). A jump-diffusion model for option pricing. *Manage. Sci.*, 48(8):1086–1101.

Meyn, S. P. and Tweedie, R. L. (1992). Stability of Markovian processes. I. Criteria for discrete-time chains. *Adv. in Appl. Probab.*, 24(3):542–574.

Nguyen, D. H. and Yin, G. (2018). Stability of regime-switching diffusion systems with discrete states belonging to a countable set. *SIAM J. Control Optim.*, 56(5):3893–3917.

Priola, E. and Wang, F.-Y. (2006). Gradient estimates for diffusion semigroups with singular coefficients. *J. Funct. Anal.*, 236(1):244–264.

Shao, J. and Xi, F. (2014). Stability and recurrence of regime-switching diffusion processes. *SIAM J. Control Optim.*, 52(6):3496–3516.

Tankov, P. (2011). Pricing and hedging in exponential Lévy models: review of recent results. In *Paris-Princeton Lectures on Mathematical Finance 2010*, volume 2003 of *Lecture Notes in Math.*, pages 319–359. Springer, Berlin.

Tuong, T. D., Nguyen, D. H., Dieu, N. T., and Tran, K. (2019). Extinction and permanence in a stochastic SIRS model in regime-switching with general incidence rate. *Nonlinear Anal. Hybrid Syst.*, 34:121–130.

Zhang, Q. (2001). Stock trading: an optimal selling rule. *SIAM J. Control Optim.*, 40(1):64–87.